EXERCISE – IV ADVANCED SUBJECTIVE QUESTIONS

1. If α & β are any two complex numbers, prove that

(i) $|\alpha+\beta|^2 + |\alpha-\beta|^2 = 2(|\alpha|^2 + |\beta|^2)$

(ii) $\left|\alpha-\sqrt{\alpha^2-\beta^2}\right| + \left|\alpha+\sqrt{\alpha^2-\beta^2}\right| = |\alpha+\beta| + |\alpha-\beta|$.

2. (a) $(1 + w)^7 = A + Bw$ where w is the imaginary cube root of of a unity and $A, B \in R$, find the ordered pair (A, B).

(b) The value of the expression ;
$1 . (2 - w)(2 - w^2) + 2. (3 - w)(3 - w^2) + \ldots\ldots$ $\ldots\ldots.. + (n - 1) . (n - w)(n - w^2)$, where w is an imaginary cube root of unity is

3. (a) Let Z is complex satisfying the equation, $z^2 - (3 + i)z + m + 2i = 0$, where $m \in R$. Suppose the equation has a real root, then find the value of m.

(b) a, b, c are real numbers in the polynomial, $P(Z) = 2Z^4 + aZ^3 + bZ^2 + cZ + 3$. It two roots of the equation $P(Z) = 0$ are 2 and i, then find the value of 'a'.

4. Find the modulus , argument and the principal argument of the complex numbers.

(i) $z = 1 + \cos\left(\frac{10\pi}{9}\right) + i\sin\left(\frac{10\pi}{9}\right)$

(ii) $(\tan 1 - i)^2$

(iii) $z = \dfrac{\sqrt{5+12i}+\sqrt{5-12i}}{\sqrt{5+12i}-\sqrt{5-12i}}$

(iv) $\dfrac{i1}{i\left(1-\cos\frac{2\pi}{2}\right)+\sin\frac{2\pi}{2}}$

5. Show that the sum $\sum_{k=1}^{2n}\left(\sin\frac{2\pi k}{2n+1} - i\cos\frac{2\pi k}{2n+1}\right)$ simplifies to a pure imaginary number.

6. Show that the product, $\left[1+\left(\frac{1+i}{2}\right)\right]\left[1+\left(\frac{1+i}{2}\right)^2\right]\left[1+\left(\frac{1+i}{2}\right)^{2^n}\right]$ is equal to $\left(1-\frac{1}{2^{2^n}}\right)(1 + i)$ where $n \geq 2$.

7. Interpret the following locii in $z \in C$.

(a) $\operatorname{Re}\left(\frac{z+2i}{iz+2}\right) \leq 4 \; (z \neq 2i)$

(b) $\operatorname{Arg}(z + i) - \operatorname{Arg}(z - i) = \pi/2$

8. Prove that the complex numbers z_1 and z_2 and the origin form an isosceles triangle with vertical angle $2\pi/3$ if $z_1^2 + z_2^2 + z_1z_2 = 0$

9. If the complex number P(w) lies on the standard unit circle in an Argand's plane and $z=(aw+b)(w-c)^{-1}$ then, find the locus of z and interpret it. Given a, b, c are real.

10. (a) Without expanding the determinant at any stage, find $K \in R$ such that $\begin{vmatrix} 4i & 8+i & 4+3i \\ -8+i & 16i & i \\ -4+Ki & i & 8i \end{vmatrix}$ has purely imaginary value.

(b) If A, B and C are the angles of a triangle

$$D = \begin{vmatrix} e^{-2iA} & e^{iC} & e^{iB} \\ e^{iC} & e^{-2iB} & e^{iA} \\ e^{iB} & e^{iA} & e^{-2iC} \end{vmatrix} \text{ where } i = \sqrt{-1}$$

then find the value of D.

11. If w is an imaginary cube root of unity then prove that

(a) $(1 - w + w^2)(1 - w^2 + w^4)(1 - w^4 + w^8) \ldots\ldots$ to 2n factors $= 2^{2n}$.

(b) If w is a complex cube root of unity , find the value of $(1 + w)(1 + w^2)(1 + w^4)(1 + w^8) \ldots\ldots$ to n factors.

12. Prove that

$$\left(\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n = \cos\left(\frac{n\pi}{2}-n\theta\right) + i\sin\left(\frac{n\pi}{2}-n\theta\right)$$

Hence deduce that

$$\left(1+\sin\frac{\pi}{5}+i\cos\frac{\pi}{5}\right)^5 + i\left(1+\sin\frac{\pi}{5}-i\cos\frac{\pi}{5}\right)^5 = 0$$

13. (a) Let z = x + iy be a complex number, where x and y are real numbers. Let A and B be the sets defied by A = {z| | z | $\le$ 2} and B = {z | (1 – i)z + (1 + i) $\bar{z}$ $\ge$ 4}. Find the area of region A $\cap$ B.

(b) For all real numbers x, let the mapping $f(x) = \frac{1}{x-i}$ (where i = $\sqrt{-1}$). If there exist real number a, b, c and d for which f(a), f(b), f(c) and f(d) form a square on the complex plane. Find the area of the square.

14. If $\begin{vmatrix} p & q & r \\ q & r & p \\ r & p & q \end{vmatrix} = 0$; where p, q, r are the moduli of non–zero complex number u, v, w respectively prove that, $\arg\left(\frac{w}{v}\right) = \arg\left(\frac{w-u}{v-u}\right)^2$.

15. The equation x^3 = 9 + 46i (where i = $\sqrt{-1}$) has a solution of the form a + bi where a and b are integers. Find the value of $(a^3 + b^3)$.

16. If ω is the fifth root of 2 and x = $\omega + \omega^2$, prove that $x^5 = 10x^2 + 10x + 6$.

17. Given that, |z – 1| = 1, where 'z' is a point on the argand plane. Show that $\frac{z-2}{z}$ = i tan (arg z).

18. If the equation $(z + 1)^7 + z^7 = 0$ has roots $z_1, z_2, \ldots, z_7$, find the value of

(a) $\sum_{r=1}^{7} \mathrm{Re}(Z_r)$

(b) $\sum_{r=1}^{7} \mathrm{Im}(Z_r)$

19. Dividing f(z) by z – i, we get the remainder i and dividing it by z + i, we get the remainder 1 + i. Find the remainder upon the division of f(z) by $z^2 + 1$.

20. If a and b are positive integer such that N = $(a + ib)^3$ – 107 i is a positive integer. Find N.

21. If the biquadratic $x^4 + ax^3 + bx^2 + cx + d = 0$ (a, b, c, d $\in$ R) has 4 non real roots, two with sum 3 + 4i and the other two with product 13 + i. Find the value of 'b'.

22. C is the complex number f : C $\to$ R is defined by f(z) = $|z^3 - z + 2|$. What is the maximum value of f on the unit circle |z| = 1 ?

23. If z_1, z_2 are the roots of the equation $az^2 + bz + c = 0$, with a, b, c > 0; $2b^2 > 4ac > b^2$; $z_1 \in$ third quadrant ; $z_2 \in$ second quadrant in the argand's plane then, show that $\arg\left(\frac{z_1}{z_2}\right) = 2\cos^{-1}\left(\frac{b^2}{4ac}\right)^{1/2}$

24. Find the set of points on the argand plane for which the real part of the complex number $(1 + i)z^2$ is positive where z = x + iy, x , y $\in$ R and i = $\sqrt{-1}$.

25. If Z_r, r = 1, 2, 3, ... 2m, m ε N are roots of the equation $Z^{2m} + Z^{2m-1} + Z^{2m-2} + \ldots + Z + 1 = 0$ then prove that $\sum_{r=1}^{2m} \frac{1}{Z_r - 1} = -m$

26. Show that all the roots of the equation $\left(\frac{1+ix}{1-ix}\right)^n = \frac{1+ia}{1-ia}$ a $\in$ R are real and distinct.